

Towards a Statistical Geometrodynamics

Ariel Caticha

Department of Physics, University at Albany–SUNY, Albany, NY 12222, USA

Abstract

Can the spatial distance between two identical particles be explained in terms of the extent that one can be distinguished from the other? Is the geometry of space a macroscopic manifestation of an underlying microscopic statistical structure? Is geometrodynamics derivable from general principles of inductive inference? Tentative answers are suggested by a model of geometrodynamics based on the statistical concepts of entropy, information geometry, and entropic dynamics.

1 Introduction

The purpose of dynamical theories is to predict or explain the changes observed in physical systems on the basis of information that is codified into what one calls the states of the system. One common view is that these dynamical theories – the laws of physics – are successful because they happen to reflect the true laws of nature.

Here I wish to follow an alternative path: perhaps once the relevant information has been identified the question of predicting changes is just a matter of careful consistent manipulation of the available information. If this turns out to be the case, then the laws of physics should follow directly from rules for processing information, that is, the rules of probability theory [1] and the method of maximum entropy (ME) [2]–[4].¹

There are some indications that this point of view is worth pursuing. Indeed, thermodynamics is a prime example of a fundamental physical theory that can be derived from general principles of inference [2]. Quantum mechanics provides a second, less trivial, and less well known example [6]. Both theories follow from a correct specification of the subject matter, that is, an appropriate choice

¹On terminology: The ME method is designed for processing information to update from a prior probability distribution to a posterior distribution. (The terms ‘prior’ and ‘posterior’ are used with similar meanings in the context of Bayes’ theorem.) The ME method is usually understood in the restricted sense that one updates from a prior distribution that happens to be uniform – this is the usual postulate of equal a priori probabilities. Here we adopt a broader meaning that includes updates from arbitrary priors and which involves the maximization of *relative* entropy. Since all entropies are relative to some prior, be it uniform or not, the qualifier ‘relative’ is redundant and will henceforth be omitted. For a brief account of the ME method in a form that is convenient for our current purposes see [5].

of variables – this is the truly difficult step – plus probabilistic and entropic arguments.

A third independent clue is found when one attempts to derive classical dynamical theories from purely entropic arguments. The surprising outcome is that the resulting “entropic” dynamics (ED) shows remarkable similarities with the general theory of relativity – geometrodynamics (GD). The general purpose of this paper is to take the first tentative steps towards explaining geometrodynamics as a form of entropic dynamics.

The procedure to derive an ED involves three steps [7]. The first step is to identify the subject matter and the corresponding space of observable states or, perhaps more appropriately, the space of macrostates. This is not easy because there exists no systematic way to search for the right macrovariables; it is a matter of taste and intuition, trial and error.

The second step is to define a quantitative measure of the change or the “distance” from one state to another. Although in general the choice of distance is not unique an exception occurs when the macrostates can be interpreted as probability distributions over some appropriate space of microstates. Then there is a natural distance which is given by the Fisher-Rao information metric [8][9] (its uniqueness is discussed in [10][11]; for a brief heuristic derivation see [12]). It measures the extent to which one probability distribution can be distinguished from another. This second step – assigning a statistical distance – is not straightforward either: more inspired guesswork is needed unless the right microstates happen to be known beforehand.²

The third and final step is easier. We ask: Given the initial and the final states, what trajectory is the system expected to follow? The question implicitly assumes that there is a trajectory, that in moving from one state to another the system will pass through a continuous set of intermediate states, and that information about the initial and final states is sufficient to determine them. The answer follows from a principle of inference, the ME principle, and not from any additional “physical” postulates.

The resulting ED is elegant and not trivial: the system moves along a geodesic but the geometry of the space of states is curved and possibly quite complicated. Since the only available clock is the system itself there is no reference to an external physical time. The natural intrinsic time is defined by the change of the system itself – in ED time is change – and can only be obtained after the equations of motion are solved. ED is a timeless Machian dynamics and its features resemble those advocated by Barbour [19]: it is reversible; it can be derived from a Jacobi action principle rather than the more familiar action principle of Hamilton; and its canonical Hamiltonian formulation is an example of a dynamics driven by constraints.

The similarities to GD are striking. For example, in GD there is no reference to an external physical time. The proper time interval along any curve between an initial and a final three-dimensional geometries of space is determined only

²The recognition that spaces of probability distributions are metric spaces has nevertheless been fruitful in statistics, where the subject is known as Information Geometry [13][14], and in physics [15]–[18].

after solving the Einstein equations of motion [20]. The absence of an external time has been a serious impediment in understanding GD because it is not clear which variables represent the true gravitational degrees of freedom [21]–[24]. GD is also derived from a Jacobi action principle [25][26] and its canonical Hamiltonian formulation is an example of a dynamics driven by constraints [27]–[29]. The question, therefore, is whether GD is an example of ED. The answer requires identifying those variables that describe the true degrees of freedom of the gravitational field.

The tentative steps of making assumptions about the subject matter, the macrostates, and about how to associate a probability distribution to each of them are taken in section 2. We want to predict the evolution of the three-dimensional geometry of space. The problem is that space is invisible. What we see is not space, but matter in space and we do not quite know how to disentangle which properties should be attributed to the matter and which to space. The best one can do is to choose the simplest form of matter: a substance that is neutral to all interactions and is itself describable by a minimal number of attributes. This ideal form of matter is a dust of identical particles; being neutral they will only interact gravitationally, and being identical the issue of what it is that distinguishes them – size, mass, flavor – does not arise. Thus we assume there is nothing to space beyond what can be learned from observing the evolving distribution of dust particles. The geometry of space is just the geometry of all the distances between dust particles. Furthermore, we assume this geometry is of statistical origin. Identical particles that are close together are easy to confuse, those that are far apart are easy to distinguish. The distance between two neighboring particles is the distinguishability distance given by a Fisher-Rao metric. Notice that the Fisher-Rao metric is used in two conceptually different ways. One is to distinguish successive states of the same system, the other is to distinguish different neighboring particles. The first is related to time, the second to space.

Having decided what system is under study and how it is statistically described we can proceed to define its ED. In section 3, as a warm up problem, we develop the ED of a single point, and then, in section 4, we generalize to the whole dust cloud. Although the resulting statistical GD is not Einstein’s GD of space-time – an indication that the states and variables we have chosen do not accurately describe the gravitational degrees of freedom – it is close enough to be encouraging. The model GD developed here corresponds to what is called an ultralocal or strong gravity theory [30]–[32]. We do not recover the notion of space-time but we do find an embryonic form of Lorentz invariance in that simultaneity is relative. Finally, in section 5 we summarize our conclusions.

2 The Geometry of a Dust Cloud

Consider a cloud of identical specks of dust suspended in an otherwise empty space. And there is nothing else; in particular, there are no rulers and no clocks, just dust. Our goal is to study how the cloud evolves. We do this by keeping

track of individual specks of dust.

Being identical the particles are easy to confuse. The only distinction between two of them is that one happens to be here while the other is over there. To distinguish one speck of dust from another we assign labels or coordinates to each particle. We assume that three real numbers (y^1, y^2, y^3) are sufficient.

Of course, particles can be mislabeled. Then the “true” coordinates y are unknown and one can only provide an estimate, x . Let $p(y|x)dy$ be the probability that the particle labeled x should have been labeled y . The labels x are introduced to distinguish one particle from another, but can we distinguish a particle at x from another at $x + dx$? If dx is small enough the corresponding probability distributions $p(y|x)$ and $p(y|x + dx)$ overlap considerably and it is easy to confuse them. We seek a quantitative measure of the extent to which these two distributions can be distinguished.

The following crude argument is intuitively appealing. Consider the relative difference,

$$\frac{p(y|x + dx) - p(y|x)}{p(y|x)} = \frac{\partial \log p(y|x)}{\partial x^i} dx^i. \quad (1)$$

The expected value of this relative difference does not provide us with the desired measure of distinguishability: it vanishes identically. However, the variance does not vanish,

$$d\lambda^2 = \int d^3y p(y|x) \frac{\partial \log p(y|x)}{\partial x^i} \frac{\partial \log p(y|x)}{\partial x^j} dx^i dx^j \stackrel{\text{def}}{=} \gamma_{ij}(x) dx^i dx^j. \quad (2)$$

This is the measure of distinguishability we seek. Except for an overall multiplicative constant, the Fisher-Rao metric γ_{ij} is the only Riemannian metric that adequately reflects the underlying statistical nature of the abstract manifold of the distributions $p(y|x)$ [10][11].

We take the further step of interpreting $d\lambda$ as the *spatial* distance of the three-dimensional space the dust inhabits. Indeed, one would normally say that the reason it is easy to confuse two particles is that they happen to be too close together. We argue in the opposite direction and *explain* that the reason the particles at x and at $x + dx$ are close together is *because* they are difficult to distinguish.

The origin of the uncertainty will be left unspecified; perhaps it is due to a limit on the ultimate resolution of observation devices, or perhaps, as with a particle undergoing Brownian motion, the uncertainty might be caused by a fluctuating physical agent. It is required, however, that two particles at the same location in space must be affected by the same uncertainty, the same irreducible noise. Then the noise is not linked to the particle, but to the place, and we might as well say that the source of the irreducible noise is space itself. This is somewhat analogous to the principle of equivalence: it is the fact that all particles irrespective of their mass move along the same trajectories in a gravitational field that allows us to eliminate the notion of a gravitational field and attribute their common behavior to a single universal agent, the curvature of space-time.

To assign an explicit $p(y|x)$ and explore the geometry it induces we will consider what is perhaps the simplest possibility. We assume that the uncertainty in the coordinate x is small so that $p(y|x)$ is sharply localized in a neighborhood about x and within this very small region curvature effects can be neglected. Further, we assume that particles are labeled by the expected values $\langle y^i \rangle = x^i$ and that the information that happens to be necessary for the purpose of prediction of future behavior is given by the second moments $\langle (y^i - x^i)(y^j - x^j) \rangle = C^{ij}(x)$. This is physically reasonable: for each particle we have estimates for its position and of the small margin of error. Then $p(y|x)$ can be determined maximizing entropy relative to an appropriate prior. To the extent that curvature effects are negligible, the underlying space is flat and translationally invariant. Thus, symmetry suggests a uniform prior and the resulting ME distribution is Gaussian,

$$p(y|x) = \frac{C^{1/2}}{(2\pi)^{3/2}} \exp \left[-\frac{1}{2} C_{ij} (y^i - x^i)(y^j - x^j) \right], \quad (3)$$

where C_{ij} is the inverse of the covariance coefficients C^{ij} , $C^{ik}C_{kj} = \delta_j^i$, and $C \equiv \det C_{ij}$. The corresponding metric is obtained substituting into eq.(2). For small uncertainties $C_{ij}(x)$ is constant within the region where $p(y|x)$ is appreciable and the result is

$$\gamma_{ij}(x) = C_{ij}(x). \quad (4)$$

The metric changes smoothly over space and, in general, space is curved. The connection, the curvature, and other aspects of its Riemannian geometry can be computed in the standard way. The probability distributions,

$$p(y|x) = \frac{\gamma^{1/2}(x)}{(2\pi)^{3/2}} \exp \left[-\frac{1}{2} \gamma_{ij}(x)(y^i - x^i)(y^j - x^j) \right], \quad (5)$$

also vary smoothly with x .

To summarize, we have succeeded in describing the information geometry that derives from considerations of distinguishability among particles. The idea is rather general but was developed explicitly only for the special case of small uncertainties, that is, for particles that can be localized within regions much smaller than those where curvature effects become appreciable. An interesting question that will not be addressed here concerns the extension to those situations of extreme curvature found near singularities.

Before discussing dynamics we mention that there is one very peculiar feature of the distance $d\lambda$, eq.(2), that may be very significant: $d\lambda^2$ is dimensionless. The metric $\gamma_{ij}(x)$ allows one to measure spatial lengths in terms of a local standard, the local uncertainty width. This immediately raises the question of how to compare the uncertainty widths, and therefore lengths, at two distant locations. One possibility, which we pursue in the rest of this paper, is that γ_{ij} describes the Riemannian geometry of space. This amounts to asserting that the uncertainty widths are the same everywhere, they provide us with a universal

standard of length. A second, more intriguing possibility, which we will explore elsewhere, is that all the information metric γ_{ij} allows us to do is to compare the lengths of small segments in different orientations at the same location; it allows one to measure angles. Then γ_{ij} does not describe the geometry of space completely, it only describes its conformal geometry.

3 Entropic Dynamics of a Single Point

In this section we develop the ED of a single Gaussian distribution, an analogue of GD in zero spatial dimensions. Let Γ be the space of states. The points in Γ are Gaussian distributions with zero mean $\langle y \rangle = 0$,

$$p(y|\gamma) = \frac{\gamma^{1/2}}{(2\pi)^{3/2}} \exp\left(-\frac{1}{2}\gamma_{ij}y^iy^j\right), \quad (6)$$

where $\gamma = \det \gamma_{ij}$ and $y = (y^1, y^2, y^3)$ are points in R^3 . Whether γ denotes the matrix γ_{ij} or its determinant should, in what follows, be clear from the context. Since $\gamma_{ij} = \gamma_{ji}$ is symmetric, Γ is a six dimensional space.

The following notation is convenient: the derivative $\partial/\partial\gamma_{ij}$ of a function $F(\gamma)$ is defined so that dF takes the simple form

$$dF \stackrel{\text{def}}{=} \frac{\partial F}{\partial\gamma_{ij}} d\gamma_{ij}. \quad (7)$$

$\partial F/\partial\gamma_{ij}$ coincides with the usual partial derivative times $(1+\delta_{ij})/2$. To operate with $\partial/\partial\gamma_{ij}$ we only need to find out how it acts on γ_{kl} and on its inverse γ^{kl} . We find

$$\frac{\partial\gamma_{kl}}{\partial\gamma_{ij}} = \frac{1}{2} \left(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j \right) \stackrel{\text{def}}{=} \delta_{kl}^{ij} \quad \text{and} \quad \frac{\partial\gamma^{kl}}{\partial\gamma_{ij}} = -\frac{1}{2} \left(\gamma^{ki} \gamma^{lj} + \gamma^{kj} \gamma^{li} \right). \quad (8)$$

Note that $\delta_{kl}^{ij} \gamma_{ij} = \gamma_{kl}$ and $\delta_{kl}^{ij} \gamma^{kl} = \gamma^{ij}$. We will also need to differentiate the determinant $\gamma = \det \gamma_{ij}$,

$$d\gamma = \gamma \gamma^{ij} d\gamma_{ij} \quad \text{or} \quad \frac{\partial\gamma}{\partial\gamma_{ij}} = \gamma \gamma^{ij}. \quad (9)$$

The Fisher-Rao metric $g^{ij\,kl}$ on the space Γ is

$$g^{ij\,kl} = \int dy p(y|\gamma) \frac{\partial \log p(y|\gamma)}{\partial\gamma_{ij}} \frac{\partial \log p(y|\gamma)}{\partial\gamma_{kl}} = \frac{1}{4} \left(\gamma^{ki} \gamma^{lj} + \gamma^{kj} \gamma^{li} \right), \quad (10)$$

and its inverse metric, defined by $g^{ij\,kl} g_{kl\,mn} = \delta_{mn}^{ij}$, is

$$g_{kl\,mn} = \gamma_{km} \gamma_{ln} + \gamma_{kn} \gamma_{lm}. \quad (11)$$

Now we can tackle the dynamics. The key to the question ‘‘Given initial and final states, what trajectory is the system expected to follow?’’ lies in

the implicit assumption that there exists a continuous trajectory. This means that large changes are the result of a continuous succession of very many small changes; the problem of studying large changes is reduced to the simpler problem of studying small changes.

We want to determine the states along a short segment of the trajectory as the system moves from an initial state γ to a neighboring final state $\gamma + \Delta\gamma$. To find the intermediate states we reason that in going from the initial to the final state the system must pass through a halfway point, that is, an intermediate state that is equidistant from γ and $\gamma + \Delta\gamma$. Finding the halfway point clearly determines the trajectory: first find the halfway point, and use it to determine ‘quarter of the way’ points, and so on. But there is nothing special about halfway states. In general, we can assert that the system must pass through intermediate states γ_ω such that, having already moved a distance $d\ell$ away from the initial γ , there remains a distance $\omega d\ell$ to be covered to reach the final $\gamma + \Delta\gamma$; ω is any positive number.

The basic dynamical question can be rephrased as follows: The system is initially described by the probability distribution $p(y|\gamma)$ and we are given the new information that the system has moved to one of the neighboring states in the family $p(y|\gamma_\omega)$. Which $p(y|\gamma_\omega)$ do we select? Phrased in this way it is clear that this is precisely the kind of problem to be tackled using the ME method.¹ The selected distribution is that which maximizes the relative entropy of $p(y|\gamma_\omega)$ relative to a prior distribution p_{old} . Since in the absence of new information there is no reason to change one’s mind, when there are no constraints the selected posterior distribution should coincide with the prior distribution. Therefore the prior p_{old} is the initial state $p(y|\gamma)$. Thus, to determine the intermediate state $\gamma_\omega = \gamma + d\gamma$ one varies over $d\gamma_{ij}$ to maximize

$$\begin{aligned} S[p(y|\gamma_\omega), p(y|\gamma)] &= - \int dy p(y|\gamma + d\gamma) \log \frac{p(y|\gamma + d\gamma)}{p(y|\gamma)} \\ &= -\frac{1}{2} g^{ij\,kl} d\gamma_{ij} d\gamma_{kl} = -\frac{1}{2} d\ell^2, \end{aligned} \quad (12)$$

subject to the constraint $d\ell_f = \omega d\ell$ where

$$d\ell_f^2 = g^{ij\,kl} (\Delta\gamma_{ij} - d\gamma_{ij}) (\Delta\gamma_{kl} - d\gamma_{kl}). \quad (13)$$

Introducing a Lagrange multiplier $\lambda/2$,

$$0 = \delta \left[-\frac{1}{2} g^{ij\,kl} d\gamma_{ij} d\gamma_{kl} + \frac{\lambda}{2} (\omega^2 d\ell^2 - d\ell_f^2) \right], \quad (14)$$

then, the selected $d\gamma_{ij}$ is given by

$$d\gamma_{ij} = \chi \Delta\gamma_{ij} \quad \text{where} \quad \chi = \frac{\lambda}{1 + \lambda(1 - \omega^2)}. \quad (15)$$

Substituting $d\gamma_{ij}$ into $d\ell$ and $d\ell_f$ we get $d\ell = \chi \Delta\ell$ and $d\ell_f = (1 - \chi) \Delta\ell$, so that $\chi = (1 + \omega)^{-1}$ with $0 < \chi < 1$ and

$$d\ell + d\ell_f = \Delta\ell. \quad (16)$$

The interpretation is clear: the three states γ , γ_ω and $\gamma + \Delta\gamma$ lie on a straight line. The expected trajectory is the geodesic that passes through the given initial and final states.

Note that each different value of ω provides a different criterion to select the trajectory and an inconsistency would arise if these criteria led to different trajectories. It is reassuring to find that indeed the ED trajectory is independent of the value ω .

ED determines the vector tangent to the trajectory $d\gamma/d\ell$, but not the actual velocity $d\gamma/dt$. In conventional forms of dynamics the distance ℓ along the trajectory is related to an external time t through a Hamiltonian which fixes the evolution relative to external clocks. But here the only clock available is the system itself which can only provide an internal, intrinsic time. It is best to define the intrinsic time so that motion looks simple. A natural definition consists in stipulating that the system moves with unit velocity, then the intrinsic time is given by the distance ℓ itself. The intrinsic time interval is the amount of change. A peculiar feature of this notion of time is that intervals are not a priori known, they are determined only after the equations of motion are solved and the actual trajectory is determined.

The geodesics in the space Γ are obtained minimizing the Jacobi action

$$J[\gamma] = \int_{\eta_i}^{\eta_f} d\eta L(\gamma, \dot{\gamma}), \quad (17)$$

where η is an arbitrary parameter along the trajectory and $\dot{\gamma}_{ij} = d\gamma_{ij}/d\eta$. The Lagrangian is just the arc length

$$L(\gamma, \dot{\gamma}) = (g^{ij\,kl} \dot{\gamma}_{ij} \dot{\gamma}_{kl})^{1/2} = \left(\frac{1}{2} \gamma^{ik} \gamma^{jl} \dot{\gamma}_{ij} \dot{\gamma}_{kl} \right)^{1/2}. \quad (18)$$

The canonical momenta are

$$\pi^{mn} = \frac{\partial L}{\partial \dot{\gamma}_{mn}} = \frac{1}{2L} \gamma^{ik} \gamma^{jl} \dot{\gamma}_{ij} \delta_{kl}^{mn} = \frac{1}{2L} \gamma^{mi} \gamma^{nj} \dot{\gamma}_{ij}, \quad (19)$$

and have a fixed magnitude

$$g_{ij\,kl} \pi^{ij} \pi^{kl} = 1. \quad (20)$$

The canonical Hamiltonian vanishes identically,

$$H_{\text{can}}(\gamma, \pi) = \dot{\gamma}_{ij} \pi^{ij} - L(\gamma, \dot{\gamma}) \equiv 0, \quad (21)$$

because the Lagrangian is homogeneous of first degree in the velocities. The manifest reparametrization invariance of the action $J[\gamma]$ conveniently reflects the absence of an external time with respect to which the system could possibly evolve.

Since variations of the momenta are constrained to preserve their magnitude the action principle is

$$I[\gamma, \pi, N] = \int_{\eta_i}^{\eta_f} d\eta [\dot{\gamma}_{ij} \pi^{ij} - N(\eta) h(\gamma, \pi)], \quad (22)$$

where

$$h(\gamma, \pi) \stackrel{\text{def}}{=} \frac{1}{2} g_{ij}{}_{kl} \pi^{ij} \pi^{kl} - \frac{1}{2}, \quad (23)$$

and $N(\eta)$ are Lagrange multipliers that at each instant η enforce the constraints

$$h(\gamma, \pi) = 0. \quad (24)$$

Equations of motion are obtained varying with respect to γ and π with γ fixed at the endpoints $\delta\gamma_{ij}(\eta_i) = \delta\gamma_{ij}(\eta_f) = 0$. Then

$$\dot{\gamma}_{mn} = N \frac{\partial h}{\partial \pi^{mn}} = 2N \gamma_{mi} \gamma_{nj} \pi^{ij} \quad (25)$$

$$\dot{\pi}^{mn} = -N \frac{\partial h}{\partial \gamma_{mn}} = -2N \gamma_{ij} \pi^{mi} \pi^{nj}. \quad (26)$$

There is no equation of motion for N . Comparing eq.(19) and (25) we get

$$N(\eta) = L(\gamma, \dot{\gamma}) = \frac{d\ell}{d\eta}, \quad (27)$$

which is recognized as the “lapse” function which gives the increase of intrinsic time ℓ per unit increase of the parameter η . Then the equations of motion simplify to

$$\frac{d\gamma_{mn}}{d\ell} = \frac{\partial h}{\partial \pi^{mn}} = 2\gamma_{mi} \gamma_{nj} \pi^{ij} \quad (28)$$

$$\frac{d\pi^{mn}}{d\ell} = -\frac{\partial h}{\partial \gamma_{mn}} = -2\gamma_{ij} \pi^{mi} \pi^{nj}. \quad (29)$$

One can check that $dh/d\eta = 0$. Therefore if $h = 0$ initially, the constraint will be consistently preserved by the evolution. One can also check that the action $I[\gamma, \pi, N]$ is invariant under the gauge transformations

$$\delta\gamma_{mn} = \varepsilon(\eta) \frac{\partial h}{\partial \pi^{mn}}, \quad \delta\pi^{mn} = -\varepsilon(\eta) \frac{\partial h}{\partial \gamma_{mn}}, \quad \text{and} \quad \delta N = \dot{\varepsilon}(\eta) \quad (30)$$

provided $\varepsilon(\eta)$ vanishes at the end points, $\varepsilon(\eta_i) = \varepsilon(\eta_f) = 0$. The invariance $\delta I = 0$ holds for any path $\gamma(\eta)$, $\pi(\eta)$ and not just for those paths at which the action is stationary. In addition, as is evident in the action $J[\gamma]$, there is an additional invariance under global (η -independent) “conformal” transformations, $\gamma_{ij} \rightarrow \psi^4 \gamma_{ij}$. The corresponding conserved quantity is $\text{tr } \pi$. To appreciate the significance of this conserved quantity note that

$$\text{tr } \pi = \gamma^{mn} \pi_{mn} = \frac{\gamma^{mn} \dot{\gamma}_{mn}}{2N} = \frac{1}{2N\gamma} \frac{d\gamma}{d\eta} = \frac{1}{2\gamma} \frac{d\gamma}{d\tau}, \quad (31)$$

so that the determinant γ expands or contracts at a constant relative rate. In particular, if the initial velocity happens to be such that $\text{tr } \pi = 0$, then γ remains fixed at its constant initial value.

4 Geometrodynamics: the Ultralocal Case

The system we study is a *single* dust cloud. To the dust cloud we associate a probability distribution P given by a product of the distributions, eq.(5) of the individual particles,

$$\begin{aligned} P[y|\gamma] &= \prod_x p(y(x)|x, \gamma_{ij}(x)) \\ &= \left[\prod_x \frac{\gamma^{1/2}(x)}{(2\pi)^{3/2}} \right] \exp \left[-\frac{1}{2} \sum_x \gamma_{ij}(x) (y^i - x^i)(y^j - x^j) \right]. \end{aligned} \quad (32)$$

It was the necessity to quantify whether we can distinguish a test particle at x from its neighbor at $x + dx$ that led us to introduce the metric γ_{ij} in the first place. When we consider the change from an earlier state γ to a later state $\gamma + \Delta\gamma$ the distinguishability problem manifests itself yet again. Even if we had managed to distinguish a test particle at x from a neighboring test particle at $x + dx$, there is no guarantee that the particle that earlier had coordinates x will be the *same* particle that will later be found at x . Particles do not just need to be identified, they need to be re-identified. For the invisible points of space this difficulty is only exacerbated because the re-identification of points depends on the state of motion of the test particles. If we allow for the possibility of particles moving past each other we conclude that the points of space cannot be treated as enduring things. And this is precisely where the model discussed in this section becomes unrealistic: we maintain such a strict correspondence between a test particle and the point it occupies that we end up treating the individual points of space as if they were real enduring objects. A more realistic model of space should deal with *several* potentially coexisting dust clouds in relative motion.

Once a dust particle in the earlier state γ is identified with the label x , we will assume that this particle can be assigned the same label x as it evolves into the later state $\gamma + \Delta\gamma$. These are commoving coordinates. Then we can write the change $\Delta\ell$ between $P[y|\gamma + \Delta\gamma]$ and $P[y|\gamma]$, eq.(32), from their relative entropy,

$$S[\gamma + \Delta\gamma, \gamma] = - \int \left(\prod_x dy(x) \right) P[y|\gamma + \Delta\gamma] \log \frac{P[y|\gamma + \Delta\gamma]}{P[y|\gamma]} = -\frac{1}{2} \Delta\ell^2 \quad (33)$$

Since $P[y|\gamma]$ and $P[y|\gamma + \Delta\gamma]$ are products $S[\gamma + \Delta\gamma, \gamma]$ can be written as a sum over the individual particles,

$$S[\gamma + \Delta\gamma, \gamma] = \sum_x S[\gamma(x) + \Delta\gamma(x), \gamma(x)] = -\frac{1}{2} \sum_x \Delta\ell^2(x), \quad (34)$$

where

$$\Delta\ell^2(x) = g^{ij\,kl}(x) \Delta\gamma_{ij}(x) \Delta\gamma_{kl}(x), \quad (35)$$

with $g^{ij\,kl}$ given by eq.(10). Therefore, the overall change in going from γ to $\gamma + \Delta\gamma$ is

$$\Delta\ell^2 = \sum_x \Delta\ell^2(x) = \int dx \rho(x) \Delta\ell^2(x), \quad (36)$$

where we have written the discrete sum as an integral – the number of dust particles within dx is $dx\rho(x)$.

Having given a sufficient specification of what we mean by a state of the system we can now proceed to formulate its ED. Once again we ask, ‘Given initial and final states, what trajectory is the system expected to follow?’ and the answer follows from the implicit assumption that there exists a continuous trajectory, but here we must pay closer attention to what precisely we mean by ‘trajectory’. Indeed, if predicting changes is just a matter of careful consistent manipulation of the available information, then we must recognize that we know more than just that the product state eq.(32) must evolve through a continuous sequence of intermediate states. We also know that each and every one of the individual factors $p(y|x, \gamma)$ must also evolve continuously through a sequence of intermediate states to reach the corresponding final state. This means that instead of one parameter ω there are many such parameters, one for each position x , and there is no reason why they should all take the same value. In other words, the intermediate states γ_ω should be labeled by a local function $\omega(x)$ rather than a single global parameter ω . A continuous sequence of states γ_ω interpolating between the initial γ and the final $\gamma + \Delta\gamma$ can be defined by imposing $\omega(x) = \zeta f(x)$ where $f(x)$ is a fixed positive function and the parameter ζ varies from 0 to ∞ . There is no single trajectory; each choice of the function $f(x)$ defines one possible trajectory. In a sense, the cloud follows many alternative paths “simultaneously”. To guarantee consistency we should check that physical predictions are independent of the choice of the arbitrary function $f(x)$.

Before we formulate the ED we should remark on the significance of invariance under choices of $f(x)$. The product state $P[y|\gamma]$ provides the only definition of what an instant is, of which states $p(y|x', \gamma')$ at distant points x' we can agree to call simultaneous with a certain state $p(y|x, \gamma)$ at the point x . Therefore, if there is no unique sequence of intermediate states, then there is no unique, absolute definition of simultaneity. We see here a kind of foliation invariance, a rudimentary, and yet extreme form of local Lorentz invariance. Since the metric γ_ω of the intermediate states $P[y|\gamma_\omega]$ remains positive for arbitrary choices of the function $\omega(x)$ the analogues of the light cones are collapsed into light lines. The invariant speed – the speed of light – is zero. The GD model described here resembles the so-called ultralocal or strong gravity theories [30]–[32] more closely than it resembles general relativity.

Now we address the question: Given initial and final states, γ and $\gamma + \Delta\gamma$, what are the possible trajectories? Let η be an arbitrary time parameter labeling successive intermediate states. The initial state is $\gamma_{ij}(\eta, x) = \gamma_{ij}(x)$, the final state is $\gamma_{ij}(\eta + \Delta\eta, x) = \gamma_{ij}(x) + \Delta\gamma_{ij}(x)$, and the intermediate states are $\gamma_{ij}(\eta + d\eta, x) = \gamma_{ij}(x) + d\gamma_{ij}(x)$. To determine the intermediate state $\gamma + d\gamma$

one varies over $d\gamma_{ij}$ to maximize the entropy

$$S[\gamma + d\gamma, \gamma] = - \int \left(\prod_x dy(x) \right) P[y|\gamma + d\gamma] \log \frac{P[y|\gamma + d\gamma]}{P[y|\gamma]} = -\frac{1}{2} d\ell^2, \quad (37)$$

where

$$d\ell^2 = \int dx \rho(x) d\ell^2(x) \quad \text{with} \quad d\ell^2(x) = g^{ij\,kl}(x) d\gamma_{ij}(x) d\gamma_{kl}(x), \quad (38)$$

subject to independent constraints at each point x ,

$$d\ell_f(x) = \omega(x) d\ell(x) \quad (39)$$

where

$$d\ell_f^2(x) = g^{ij\,kl}(x) (\Delta\gamma_{ij}(x) - d\gamma_{ij}(x)) (\Delta\gamma_{kl}(x) - d\gamma_{kl}(x)) \quad (40)$$

Introducing Lagrange multipliers $\lambda(x)/2$,

$$0 = \delta \left[\int dx \rho(x) \left\{ -\frac{1}{2} d\ell^2(x) + \frac{\lambda(x)}{2} (\omega^2(x) d\ell^2(x) - d\ell_f^2(x)) \right\} \right] \quad (41)$$

the result, $d\gamma_{ij}(x) = \chi(x) \Delta\gamma_{ij}(x)$, coincides with the single point result, eq.(15) for each value of x . Substituting $d\gamma_{ij}$ into $d\ell(x)$ and $d\ell_f(x)$ we get $d\ell(x) = \chi \Delta\ell(x)$ and $d\ell_f(x) = [1 - \chi(x)] \Delta\ell(x)$, so that

$$d\ell(x) + d\ell_f(x) = \Delta\ell(x). \quad (42)$$

The conclusion is that the states of the individual particles evolve independently of each other along geodesics in the single point configuration space given by eqs.(28-29). The dynamics of the cloud is independent of the choice of $\omega(x)$ as desired – this is foliation invariance.

The ultralocal statistical GD deduced in the previous paragraphs is the dynamics of a large or perhaps infinite number of independent subsystems. The action for the whole cloud can be written as the sum of the individual particle actions given in eq.(17). Thus, the proposed action is

$$J[\gamma, \dot{\gamma}] = \int_{\eta_i}^{\eta_f} d\eta \int dx \rho (g^{ij\,kl} \dot{\gamma}_{ij} \dot{\gamma}_{kl})^{1/2}, \quad (43)$$

where $\dot{\gamma}_{ij} = \partial\gamma_{ij}/\partial\eta$. In commoving coordinates $\dot{\rho} = \partial\rho/\partial\eta = 0$. It is straightforward to develop the constrained Hamiltonian formalism and recover the single particle equations of motion.

Notice that the actual distance from the initial state to the final state along a certain path is given by eq.(36),

$$\ell = \int_i^f d\eta (\dot{\ell}^2)^{1/2} = \int_{\eta_i}^{\eta_f} d\eta \left[\int dx \rho g^{ij\,kl} \dot{\gamma}_{ij} \dot{\gamma}_{kl} \right]^{1/2}. \quad (44)$$

Therefore, unlike the action for a single point, eq.(17), the action (43) is not the natural arc length. The dust cloud does not evolve along a geodesic. The reason for this can be traced to the additional constraint that individual particles evolve continuously, which allows a multitude of different trajectories and leads to foliation invariance.

5 Conclusions

One idea explored in this work is whether it is possible to establish a connection between ordinary spatial distances and the information metric of Fisher and Rao – whether one can explain the notion of spatial distance. We succeeded in describing the information geometry that derives from considerations of distinguishability among particles; particles that are easily confused are said to be near, those that are easily distinguished are farther apart. The idea is that distances between particles are not distances between structureless points but distances between probability distributions.

According to Euclid, a point is that which has no size. General relativity was founded upon a revision of Euclid’s fifth postulate. Statistical geometrodynamics is founded upon the further revision of Euclid’s first definition, the notion of structureless points.

The second idea we explored is whether Einsteinian macroscopic geometrodynamics is derivable from an underlying microscopic statistical theory purely on the basis of principles of inference, without additional postulates of a more “physical” nature. We can only claim a partial success; the result is close enough to be promising. The model GD we obtained satisfies the main requirement, it describes the dynamics of a geometry; it is related to gravity because it describes an ultralocal gravity theory; and it exhibits foliation invariance. Moreover, the somewhat puzzling fact that space and time are so different and yet enter the formalism in such a symmetric way receives a natural explanation: a time interval refers to the extent we can distinguish an earlier state from a later state of the same system, while a spatial distance refers to the extent we can distinguish two different systems.

Einstein’s GD might be recovered by making a different choice of the states and variables that describe the gravitational degrees of freedom. Two possible alternative choices were suggested. First, one should avoid a too strict correspondence between a test particle and the point it occupies because this treats the individual points of space as if they were real objects. Second, it may be that the Fisher-Rao metric does not describe the full geometry of space, as we assumed in this work, but only describes its conformal geometry.

Should the ideas proposed here prove successful one can further expect that the currently popular approaches to a quantum theory of gravity will require revision.

References

- [1] R. T. Cox: Am. J. Phys. **14**, 1 (1946); *The Algebra of Probable Inference* (Johns Hopkins, Baltimore, 1961).
- [2] E. T. Jaynes: Phys. Rev. **106**, 620 and **108**, 171 (1957); *E. T. Jaynes: Papers on Probability, Statistics and Statistical Physics*, ed. by R. D. Rosenkrantz (Reidel, Dordrecht, 1983).

- [3] J. E. Shore and R. W. Johnson: IEEE Trans. Inf. Theory **IT-26**, 26 (1980).
- [4] J. Skilling: ‘The Axioms of Maximum Entropy’. In: *Maximum-Entropy and Bayesian Methods in Science and Engineering*, ed. by G. J. Erickson and C. R. Smith (Kluwer, Dordrecht, 1988).
- [5] A. Caticha: ‘Maximum Entropy, Fluctuations and Priors’. In: *Bayesian Methods and Maximum Entropy in Science and Engineering*, ed. by A. Mohammad-Djafari, AIP Conf. Proc. **568**, 94 (2001) (arXiv.org/abs/math-ph/0008017).
- [6] A. Caticha: Phys. Lett. **A244**, 13 (1998); Phys. Rev. **A57**, 1572 (1998); Found. Phys. **30**, 227 (2000) (arXiv.org/abs/quant-ph/9810074).
- [7] A. Caticha: ‘Entropic Dynamics’. In: *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by R. L. Fry, AIP Conf. Proc. **617**, 302 (2002). (arXiv.org/abs/gr-qc/0109068).
- [8] R. A. Fisher: Proc. Cambridge Philos. Soc. **122**, 700 (1925).
- [9] C. R. Rao: Bull. Calcutta Math. Soc. **37**, 81 (1945).
- [10] N. N. Čencov: *Statistical Decision Rules and Optimal Inference*, Transl. Math. Monographs, vol. 53, Am. Math. Soc. (Providence, 1981).
- [11] L. L. Campbell: Proc. Am. Math. Soc. **98**, 135 (1986).
- [12] A. Caticha: ‘Change, Time and Information Geometry’. In: *Bayesian Methods and Maximum Entropy in Science and Engineering*, ed. by A. Mohammad-Djafari, AIP Conf. Proc. **568**, 72 (2001) (arXiv.org/abs/math-ph/0008018).
- [13] S. Amari: *Differential-Geometrical Methods in Statistics* (Springer-Verlag, 1985).
- [14] C. C. Rodríguez: ‘Entropic Priors for Discrete Probabilistic Networks and for Mixtures of Gaussian Models’. In: *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by R. L. Fry, AIP Conf. Proc. **617**, 410 (2002).
- [15] F. Weinhold: J. Chem. Phys. **63**, 2479 (1975).
- [16] G. Ruppeiner: Phys. Rev. A **20**, 1608 (1979) and **27**, 1116 (1983).
- [17] L. Diósi and B. Lukács: Phys. Rev. A **31**, 3415 (1985) and Phys. Lett. **112A**, 13 (1985).
- [18] R. Balian, Y. Alhassid and H. Reinhardt: Phys. Rep. **131**, 1 (1986).
- [19] J. Barbour: Class. Quant. Grav. **11**, 2853(1994).

- [20] R. F. Baierlein, D. H. Sharp and J. A. Wheeler: Phys. Rev. **126**, 1864 (1962).
- [21] J. W. York: ‘Kinematics and Dynamics of General Relativity’. In: *Sources of Gravitational Radiation*, ed. by L. Smarr (Cambridge Univ. Press, 1979).
- [22] C. Rovelli: ‘Strings , Loops and the Others: a Critical Survey of the Present Approaches to Quantum Gravity’. In: *Gravitation and Relativity: At the Turn of the Millenium*, ed. by N. Dadhich and J. Narlikar (Poona Univ. Press, 1998).
- [23] K. Kuchar: ‘Time and Interpretations of Quantum Gravity’. In: *Proc. of the 4th Canadian Conf. on General Relativity and Astrophysics*, ed. by G. Kunsatter, D. Vincent, and J. Williams (World Scientific, Singapore, 1992).
- [24] C. Isham: ‘Canonical Quantum Gravity and the Problem of Time’. In: *Integrable Systems, Quantum Groups and Quantum Field Theories*, ed. by L. A. Ibort and M. A. Rodriguez (World Scientific, Singapore, 1993).
- [25] J. D. Brown and J. W. York: Phys. Rev. **D40**, 3312 (1989).
- [26] J. Hartle: Phys. Rev. **D38**, 2985 (1988).
- [27] P. Dirac: Proc. Roy. Soc. Lond., **A246**, 333 (1958).
- [28] R. Arnowitt, S. Deser and C. W. Missner: ‘The Dynamics of General Relativity’. In: *Gravitation: an Introduction to Current Research*, ed. by L. Witten (Wiley, New York, 1962).
- [29] S. A. Hojman, K. Kuchar and C. Teitelboim: Ann. Phys. **96**, 88 (1976).
- [30] C. J. Isham: Proc. Roy. Soc. Lond., **A351**, 209 (1976).
- [31] C. Teitelboim: Phys. Rev. **D25**, 3159 (1982).
- [32] M. Pilati: Phys. Rev. **D26**, 2645 (1982).